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Hermite Interpolation over Curved Finite Elements

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1. Introduction

The essential starting point in any finite element calculation is the choice of the approximating subspace from the space of admissible functions. In this there are two different approaches. The first is to choose a set of elements and points and to construct a basis to interpolate function value at each of these points or nodes, as we shall call them. This type of interpolation is called Lagrange interpolation. If some of the function's derivatives are also interpolated, we refer to this as Hermite interpolation. In the literature, the term "element" is often used to refer to a particular geometrical shape together with a set of basis functions. However, since many different sets of basis functions can be used with the same geometrical shape, we shall refer to the shape alone as the "element" and the basis functions as a basis for a particular element. The choice of element shape will be dictated by the geometry of the problem and the accuracy required in matching any geometrical irregularities within the domain of interest. One would certainly use the simplest shape possible, and in this respect the triangle has proved very popular and has been the object of much attention in the past. Zlámal [12] introduced cubic and quintic Hermite bases for the triangle in 1968, and these and closely related bases have become extremely popular. One reason for the popularity of Hermite bases is the direct connection between parameters used in the method, i.e., function derivatives, and important physical concepts of the problem, e.g., energy. Unique polynomials for a wider range of Hermite interpolation (and for Lagrange interpolation) and error estimates have been given more recently by Ciarlet and Raviart [2]. As methods developed it became desirable to be able to construct basis

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functions for curved elements. The curved element was introduced into structural analysis by Ergatoudis, Irons, and Zienkiewicz [3]. The technique they employed is commonly referred to as the "isoparametric transformation" technique. It is most commonly used in its Lagrange form where the transformation depends only on the basis functions and coordinates of the nodes. In its Hermite form the transformation is of an implicit nature, certain derivatives having to be specified before the transformation is completely determined [7]. There are two aspects of the method, be it used in the Lagrange or Hermite form, of particular relevance to the present discussion. Firstly, the particular basis functions used to define the required transformation also, by implication, give an approximation to any curved edges of the element. That is, the processes of approximating the geometry of the given domain and the choice of basis functions cannot be separated. McLeod and Mitchell [6] have discussed this for certain isoparametric transformations. Secondly, the method, by the way in which it is defined, is strictly only first order in that the basis is only exact for linear polynomials. An interesting study of this was given by Bond *et al.* [1], where it was shown that serious loss of accuracy resulted when the elements were greatly distorted from their corresponding straight-edged counterparts. These authors proposed a different type of basis which was affected less by the element distortion than the isoparametric one. This basis, however, lacked the conformity of an isoparametric one. In the case of Lagrange interpolation alternatives which allowed greater freedom in the approximation of curves and which were conforming have been proposed [5, 8] and node requirements for any order of approximation over any two-dimensional element bounded by algebraic curves have been given [4]. In the present note we shall discuss the problem of producing a conforming Hermite basis for $(2n + 1)$th degree approximation. We restrict ourselves in this note to three-sided elements where each side is an arc of a conic. The reader may find a little algebraic geometry background helpful to his understanding of the arguments presented here [9–11].

2. The Dimension of the Basis

Since the $(2n + 1)$th degree polynomial in two variables has $(2n + 3)(n + 1)$ degrees of freedom we will require at least this number of basis functions. For the straight-sided triangle this number will be sufficient and the sufficiency can be seen in the following way. Conformity requires that each basis function is identically zero on all sides not containing its node. To check that this condition is satisfied on a particular side one only needs to check that the basis function satisfying the homogeneous conditions on that side is identically zero. Consider the $(2n+1)$th degree polynomial interpolating to function value and the partial derivatives up to order n at the vertices of a triangle. If this polynomial satisfies the homogeneous conditions at these points then this is equivalent to the corresponding algebraic curve having a point of multiplicity $n+1$ at each vertex. This implies that the curve has $2n+2$ points in common with the line joining any two vertices and hence, by Bezout's theorem, must have a common component. Since the line is irreducible, the polynomial must be identically zero on the line. Thus interpolation of function and partial derivatives up to order n is quite sufficient to ensure conformity. This gives a total of $\frac{3}{2}(n+2)(n+1)$ basis functions associated with nodes on the boundary of the element. This is less than $(2n+3)(n+1)$ and hence an additional $(n/2)(n+1)$ basis functions associated with interior points and interpolating function value only are taken so that the final basis can span polynomials up to degree $2n+1$. Using the same argument when the sides of the element are conics, we see that additional nodes must be placed on the sides of the element. In this case the corresponding algebraic curve to the interpolating polynomial of degree $2n+1$ and the conic arc still have $2(n+1)$ points in common, but this is now insufficient to imply that they have a common component. However, if we place an additional $2n+1$ nodes on the conic side and interpolate function value at each of these points, then the curve of degree $2n+1$ will now have $4n+3$ points in common with the conic. Again by Bezout's theorem the two curves must have a common component. Since the conic is assumed irreducible, the polynmial of degree $2n+1$ must be identically zero on the conic. Conformity will then be assured. We will also take an additional $(n/2)(n+1)$ interior points similar to the straight-sided triangle and interpolate function value at these points. The total dimension of the basis will then be

$$(2n+3)(n+1)+I(2n+1), \tag{1}$$

where I is the number of conic arcs.

3. A Basis for the Straight-sided Triangle

As outlined in the previous section, a basis of dimension $(2n+3)(n+1)$ can satisfy the condition of conformity and span all polynomials of degree less than or equal to $2n+1$. Polynomial bases satisfying these conditions are well known and are easily obtained by demanding that the general polynomial of degree $2n+1$ satisfy the appropriate interpolation conditions. It is convenient for our purpose to write such a basis in the following form.

Let the set $\{H_\alpha(x, y)\}$ ($\alpha = 1, 2, \ldots, (n/2)(n + 1)$) be the set of basis functions which satisfy the homogeneous conditions at the vertices, i.e.,

$$\left.\frac{\partial^{i+j} H_\alpha(x, y)}{\partial x^i\, \partial y^j}\right|_{V_k} = 0, \qquad i, j = 0, 1, \ldots, n, \quad i + j \leqslant n, \text{ where } V_k,\ k = 1, 2, 3, \text{ are the vertex nodes,} \tag{2}$$

and

$$H_\alpha(x_\beta, y_\beta) = \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, \ldots, (n/2)(n + 1),\ \delta_{ij} = 0,\ i \neq j,\ \delta_{ii} = 1,$$

where (x_α, y_α) are the coordinates of the node associated with $H_\alpha(x, y)$. These nodes will be chosen in the interior of the element.

Let the remaining basis functions be labeled $\{T_{\alpha\beta\gamma}(x, y)\}$ and satisfy

$$\left.\frac{\partial^{i+j} T_{\alpha\beta\gamma}(x, y)}{\partial x^i\, \partial y^j}\right|_{V_k} = \delta_{i\alpha}\delta_{j\beta}\delta_{k\gamma}, \qquad \text{for } \alpha, \beta, i, j = 0, 1, 2, \ldots, n, \quad i + j, \alpha + \beta \leqslant n, \quad k, \gamma = 1, 2, 3, \tag{3}$$

nd

$$T_{\alpha\beta\gamma}(x_\delta, y_\delta) = 0, \qquad \delta = 1, 2, \ldots, \frac{n}{2}(n + 1),$$

(i.e., $T_{\alpha\beta\gamma}(x, y)$ is zero at all the interior nodes). For example, $T_{213}(x, y)$ is zero at all the interior nodes, is zero and has zero partial derivatives up to and including order n at vertex nodes V_1 and V_2, and is zero with zero derivatives except $\partial^3 T_{213}(x, y)/\partial x^2 \partial y$ at vertex node V_3. At this node

$$\partial^3 T_{213}(x, y)/\partial x^2\, \partial y = 1.$$

The complete basis is then given by the $(2n + 3)(n + 1)$ functions $\{H_\alpha(x, y), T_{\alpha\beta\gamma}(x, y)\}$. We must note here that for uniqueness of the basis the interior nodes must be such that no three of them lie on a line, no six lie on a conic and in general no subset of $\frac{1}{2}(m + 2)(m + 1)$ of them can lie on a curve of degree m.

4. The Form of the Basis for the Curved Element

Notation. We will introduce basis functions which will be labeled $WV_{\alpha\beta\gamma}(x, y)$, $WI_i(x, y)$, and $WC_i(x, y)$. These basis functions will be functions of the independent space variables x and y. However, for ease in writing, we will usually write these as $WV_{\alpha\beta\gamma}$, WI_i, and WC_i, the dependence being understood. We will also do this with the aforementioned functions $T_{\alpha\beta\gamma}(x, y)$ and $H_i(x, y)$. We will also omit the subscript when unnecessary. Occasionally, however, it is necessary to include subscripts and the dependence, e.g.,

$H_j(x_i, y_i)$ means the particular function from the set $\{H_k(x, y)\}$ associated with node j and evaluated at the point (x_i, y_i). We hope there is no confusion.

We have now decided on our choice and position of nodes for the curved element. We have $(n/2)(n+1)$ interior nodes and intend to interpolate function value at these nodes. We have $2n+1$ nodes between vertices on each curved side and will interpolate function value at these points. Finally we have the vertex nodes and there we will interpolate function value and partial derivatives up to and including order n. We will label the corresponding basis functions $\{WI_i\}$ for the interior nodes, $\{WC_i\}$ for the nodes on the curves, and $\{WV_{\alpha\beta\gamma}\}$ for the vertex nodes. We will require that these functions satisfy similar properties to the $\{H_i\}$ and $\{T_{\alpha\beta\gamma}\}$, though we now require these properties to be satisfied at more nodes. For example, the $\{WV_{\alpha\beta\gamma}\}$ must satify identical properties to the $\{T_{\alpha\beta\gamma}\}$ at vertices but must be zero not just at the interior nodes but also at the nodes on curved sides.

We ultimately wish the basis to span polynomials up to degree $2n+1$. This gives us the $(2n+3)(n+1)$ conditions (written symbolically)

$$\begin{aligned}
\sum WV_{\alpha\beta\gamma} + \sum WI_i \quad + \sum WC_i \quad &= 1 \\
\sum \frac{\partial^{\alpha+\beta} x}{\partial x^\alpha\, \partial y^\beta}\bigg|_{V_\gamma} WV_{\alpha\beta\gamma} + \sum x_i WI_i \quad + \sum x_i WC_i \quad &= x \\
\vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots & \\
\sum \frac{\partial^{\alpha+\beta} (y^{2n+1})}{\partial x^\alpha\, \partial y^\beta}\bigg|_{V_\gamma} WV_{\alpha\beta\gamma} + \sum y_i^{2n+1} WI_i + \sum y_i^{2n+1} WC_i &= y^{2n+1}.
\end{aligned} \tag{4}$$

However, since the set $\{H_i, T_{\alpha\beta\gamma}\}$ spans these polynomials, we could equivalently demand that our basis $\{WV_{\alpha\beta\gamma}, WI_i, WC_i\}$ span the basis $\{H_i, T_{\alpha\beta\gamma}\}$. This leads to the equivalent system

$$\begin{aligned}
WI_1 + \sum H_1(x_i, y_i)\, WC_i &= H_1(x, y) \\
WI_2 + \sum H_2(x_i, y_i)\, WC_i &= H_2(x, y) \\
\vdots \qquad\qquad \vdots \qquad\quad & \quad \vdots \\
WI_N + \sum H_N(x_i, y_i)\, WC_i &= H_N(x, y), \quad \text{where} \quad N = (n/2)(n+1) \\
WV_{001} + \sum T_{001}(x_i, y_i)\, WC_i &= T_{001}(x, y) \\
WV_{002} + \sum T_{002}(x_i, y_i)\, WC_i &= T_{002}(x, y) \\
\vdots \qquad\qquad \vdots \qquad\quad & \quad \vdots \\
WV_{101} + \sum T_{101}(x_i, y_i)\, WC_i &= T_{101}(x, y) \\
\vdots \qquad\qquad \vdots \qquad\quad & \quad \vdots \\
WV_{0n3} + \sum T_{0n3}(x_i, y_i)\, WC_i &= T_{0n3}(x, y),
\end{aligned} \tag{5}$$

where we have used the properties given in Eqs. (2) and (3). This can be written in the form

$$\begin{bmatrix} 0 & 0 & . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & . & 0 \\ 1 & 0 & 0 & . & . & . & . & . \\ 0 & 1 & 0 & . & . & . & . & . \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 1 \end{bmatrix} \begin{bmatrix} WV_{001} \\ WV_{002} \\ \\ \vdots \\ \\ WV_{0n3} \end{bmatrix}$$

$$= \begin{bmatrix} H_1(x, y) - WI_1 - \sum H_1(x_i, y_i)\, WC_i \\ \vdots \\ H_N(x, y) - WI_N - \sum H_N(x_i, y_i)\, WC_i \\ T_{001}(x, y) - \sum T_{001}(x_i, y_i)\, WC_i \\ \vdots \\ T_{0n3}(x, y) - \sum T_{0n3}(x_i, y_i)\, WC_i \end{bmatrix}. \tag{6}$$

This system is inconsistent unless

$$WI_j + \sum H_j(x_i, y_j)\, WC_i = H_j(x, y), \qquad j = 1, 2, \ldots, N, \tag{7}$$

in which case the complete basis is given by $\{WI_i, WC_i, WV_{\alpha\beta\gamma}\}$, where

$$WV_{\alpha\beta\gamma}(x, y) = T_{\alpha\beta\gamma}(x, y) - \sum T_{\alpha\beta\gamma}(x_i, y_i)\, WC_i\,. \tag{8}$$

Hence if we have a suitable set of $\{WI_i, WC_i\}$ which also satisfy Eq. (7), then the basis can be completed by using Eq. (8). We will now produce such a basis.

5. Basis Construction

For ease of understanding we will work through a simple case before stating the more general results. Consider the case where we have an element with two straight sides and one conic arc, and we wish to construct a basis for cubic polynomials on this element. This corresponds to the case where $n = 1$ and $I = 1$ in Eq. (1); we will have a total of 13 basis functions, one associated with an interior node, three associated with nonvertex nodes on the conic side, and the remaining nine associated with the vertex nodes. The situation is depicted in Fig. 1, where the straight sides are 12 and 13

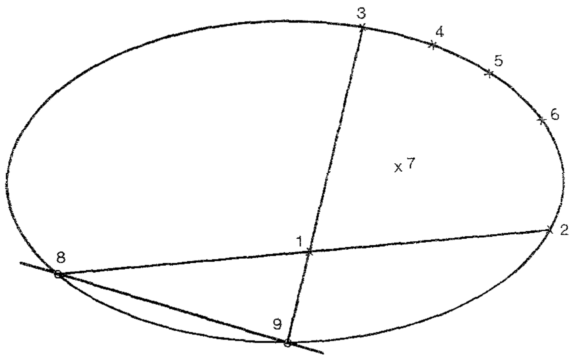

FIG. 1. Node positions for a basis for cubic polynomials on the element with two straight sides and one conic arc. This corresponds to the case $n = 1$, $I = 1$ in Eq. (1).

and the conic arc (in this case drawn as an ellipse) is 34562, and the interior node has been labeled 7. We must produce a satisfactory WI_7, WC_4, WC_5, WC_6, and then Eq. (8) will enable us to complete the basis.

Let $(A; B)$ denote the linear form defined by the line AB, and let $(A; B)_i$ denote this linear form normalized to have unit value at node i.

Similarly, let $(A; B; C; D; E)$ denote the quadratic form defined by the points A, B, C, D, E (note that in general five points uniquely determine a conic), and let $(A; B; C; D; E)_i$ denote this quadratic form normalized to have unit value at node i.

Let the conic arc be given by $f(x, y) = 0$ and let f_i be the similarly normalized quadratic form. Extend all element sides to give the set of external intersection points (in this case, only two points 8 and 9). With the notation so defined we note that

$$(1; 3)_j \equiv (1; 9)_j \equiv (3; 9)_j\,, \text{ etc.}$$

The external intersection points 8 and 9 uniquely determine a line, the linear form of which, (8; 9), we denote by d.

Consider

$$WC_4 = (1; 3)_4(1; 2)_4(3; 5; 6; 2; 7)_4/d_4\,.$$

By definition, $WC_4 = 1$ at node 4. Also $WC_4 = 0$ at nodes 1, 2, 3, 5 and 6, i.e.,

$$WC_4(j) = \delta_{4j}\,.$$

By construction we are also assured that WC_4 is identically zero on the opposite sides 12 and 13. Thus conformity across these sides will be assured. We must now show that WC_4 reduces to a cubic along the conic arc so that we will ultimately be able to use Bezout's theorem to show conformity across this side. We follow the same arguments as Wachspress [8] where, as a special case of Max Noether's fundamental theorem, we have

$$(9; 3)(8; 2) \equiv (8; 9)(3; 2) \bmod f.$$

Therefore

$$(1; 3)(1; 2)/d \equiv (2; 3) \bmod f,$$

and hence

$$WC_4 \equiv (2; 3)(3; 5; 6; 2; 7) \bmod f,$$

i.e., reduces to a cubic on f. Also, by construction WC_4 has double points at 1, 2, 3 and hence $\partial WC_4/\partial x = \partial WC_4/\partial y = 0$ at these points.

WC_4 hence has all the properties desired of it. In a similar fashion we define WC_5 and WC_6 to give the set

$$\begin{aligned} WC_4 &= (1; 3)_4(1; 2)_4(3; 5; 6; 2; 7)_4/d_4 \\ WC_5 &= (1; 3)_5(1; 2)_5(3; 4; 6; 2; 7)_5/d_5, \\ WC_6 &= (1; 3)_6(1; 2)_6(3; 4; 5; 2; 7)_6/d_6. \end{aligned} \tag{9}$$

Now let

$$WI_7 = (1; 3)_7(1; 2)_7 f_7/d_7. \tag{10}$$

By construction $WI_7 \equiv 0$ on the element sides and has double points at 1, 2, and 3. Now consider

$$F(x, y) \equiv H_7(x, y) - WI_7(x, y) - \sum_{i=4}^{6} H_7(x_i, y_i)\, WC_i(x, y). \tag{11}$$

This function is zero at 1, 2, 3, 4, 5, and 6, and has double points at 1, 2, and 3. Therefore

$$F(x, y) \equiv 0,$$

or

$$F(x, y) = \alpha(1; 3)(1; 2) f/d$$

for some scalar α. But $F(x_7, y_7) = 0$ and since (1; 3), (1; 2), and f are nonzero at node 7, $F(x, y)$ must be identically zero. Thus WI_7, WC_4, WC_5, WC_6 span H_7, i.e., Eq. (7) is satisfied and hence we can use Eq. (8) to complete the basis. This gives us a basis of 13 functions which together span all polynomials of degree less than or equal to 3.

6. The General Case

In general, we must produce a set of basis functions associated with nodes on the conic sides and a set associated with interior nodes. Each of these basis functions will be of the form

$$\kappa(O(x, y)\, N(x, y)/D(x, y), \qquad \text{where } \kappa \text{ is a normalizing constant,} \tag{12}$$

where we associate $O(x, y)$ as the term required to ensure that the basis function is identically zero on all element sides not containing the appropriate node. $N(x, y)$ will be the term which ensures that the basis function is also zero at the remaining nodes, with the obvious exception of the one node associated with the basis function. $D(x, y)$ will be the term required to ensure that the basis function reduces to the appropriate degree polynomial along the side associated with the particular basis function. We will deal with each of these terms in turn.

1. $D(x, y)$

Two algebraic curves of degrees m and n have mn intersections and hence the total number of intersections of element sides in our case will be 3, 5, 8, or 12 when we have no conic sides, 1, 2, and 3 conic sides, respectively. (For the moment we are assuming that all the intersections are simple.) Three of these intersections are vertices, which leaves us with 0, 2, 5, or 9 external intersection points. These external intersection points then uniquely determine algebraic curves of orders 0, 1, 2, or 3. The term $D(x, y)$ is taken to be the polynomial associated with the algebraic curve defined by the external intersection points of the curves defining the element. We note that this polynomial is of degree $K - 3$, where K is the sum of the degrees of the polynomials associated with the curves defining the element. We also note that this is the same denominator polynomial discussed in more detail (and in the special cases where the intersections are not simple) by Wachspress [8]. Indeed, it was Wachspress who first applied the ideas of algebraic curve intersection theory to the production of basis functions for curved elements.

2. $O(x, y)$

This term is simply the product of the polynomials associated with element sides which do not include the node being considered. Hence, for all the basis functions $\{WI\}$, the corresponding $O(x, y)$ will be the product of the three polynomials associated with the element sides. This is because the WI basis functions are the functions corresponding to interior nodes and the interior nodes do not lie on any of the element sides. For the $\{WC\}$ functions the $O(x, y)$ will be the product of the polynomials associated with the two opposite sides.

3. $N(x, y)$

For this term we will consider the $\{WI\}$ functions and the $\{WC\}$ functions separately. For degree $2n + 1$ approximation we have $(n/2)(n + 1)$ interior nodes. Now for the $\{WI\}$ functions the $O(x, y)$ terms ensure that the basis function is identically zero along the element sides and hence at all vertex and side nodes. However, we must also ensure that each of these basis functions is zero at the remaining $(n/2)(n + 1) - 1$ interior nodes. We must also ensure that each WI has a zero of multiplicity at least $n + 1$ at each of the vertices. For $n = 1$ this is assured, since the term $O(x, y)$ has a zero of multiplicity two at each vertex. However, for $n = 2$ and higher this is not the case and extra zeros at vertices must be included in the corresponding $N(x, y)$ term. Now for $n \geqslant 2$, the $N(x, y)$ term for each node must have $(n/2)(n + 1) - 1$ simple zeros at the remaining interior nodes and zeros of multiplicity $n - 1$ at each of the vertices. A zero of multiplicity $n - 1$ imposes $(n/2)(n - 1)$ linear conditions on the coefficients of a polynomial. Hence we have a total of

$$3 \times (n/2)(n - 1) + (n/2)(n + 1) - 1$$

linear conditions on the coefficients of $N(x, y)$, i.e.,

$$2n^2 - n - 1.$$

This is the exact number required to uniquely determine a curve of degree $2n - 2$. We will then define, for each interior node, $N(x, y)$ to be the polynomial which has simple zeros at the remaining $(n/2)(n + 1) - 1$ interior nodes and zeros of multiplicity $n - 1$ at each vertex. We point out here that the lines joining each pair of vertices will have $2n - 2$ intersections with each of the functions $N(x, y)$. This does not contradict Bezout's theorem. However, if any of the interior nodes lie on the line joining any two vertices, then Bezout's theorem would imply that $N(x, y)$ was reducible, having this line as a factor. In this case the remaining $n^2 - 2$ conditions are insufficient to uniquely determine the remaining factor of degree $2n - 3$. That is, for uniqueness we must ensure that the node selection is such that no subset of $m(2n - 2) + 1$ points, counting multiplicity, lies on a curve of degree m, $m \leqslant 2n - 2$.

For the basis functions associated with nodes of the conic sides the argument is similar. Here the $O(x, y)$ term ensures us that the basis function will be identically zero on opposite sides and hence at all the nodes on that side. We are also assured of a double zero at the opposite vertex and simple zeros at the adjacent vertices. The term $N(x, y)$ in this case must be zero at the remaining $2n$ nodes on the conic side and at the $(n/2)(n + 1)$ interior nodes. It must also have a zero of multiplicity $n - 1$ at the opposite vertex and zeros of multiplicity n at each of the adjacent vertices. This gives a total of $2n^2 + 3n$ linear conditions on $N(x, y)$. This is exactly the number required to uniquely determine a curve of degree $2n$. In this case then we define $N(x, y)$ to be the polynomial associated with the unique curve of degree $2n$ having simple zeros at the remaining $2n$ nodes on the conic and the $(n/2)(n + 1)$ interior nodes, a zero of multiplicity $n - 1$ at the opposite vertex, and zeros of multiplicity n at each of the adjacent vertices. We note here that the conic and the curve given by $N(x, y) = 0$ have $4n$ intersections and again there is no contradiction of Bezout's theorem.

7. Consistency

By construction, the sets $\{WI\}$ and $\{WC\}$ have unit value at the corresponding node and are zero at all other nodes. They also have zero partial derivatives up to order n at each vertex. Also, by construction, they are identically zero on all element sides not containing the corresponding node. We are therefore assured of interelement continuity, i.e., conformity. It only remains to show that these sets satisfy Eq. (7).

For any basis function WC, the $O(x, y)$ term is the product of the two opposite sides and hence is quadratic, cubic, or quartic. Similarly the $D(x, y)$ term is linear, quadratic, or cubic. Looking at each of these cases in turn we have:

1. *One Conic Side* (Fig. 2a)

$$O(x, y)/D(x, y) = K(1; 3)(2; 3)(4; 5) = K(1; 5)(2; 4)(4; 5).$$

Now the conics $(1; 5)(2; 4)$, $(4; 5)(1; 2)$, and $f(x, y)$ have a common intersection cycle in pairs, namely the points 1, 2, 4, and 5. Therefore

$$(1; 5)(2; 4) \equiv (4; 5)(1; 2) \bmod f(x, y);$$

therefore

$$(1; 5)(2; 4)/(4; 5) \equiv (1; 2) \bmod f(x, y),$$

i.e.,

$$O(x, y)/D(x, y) \equiv (1; 2) \bmod f(x, y).$$

2. *Two Conic Sides* (Fig. 2b)

$D(x, y)$ is the polynomial associated with the conic defined by points 4, 5, 6, 7, and 8. For nodes on the conic $f(x, y) = 0$, $O(x, y) = g(x, y)(2; 7)$.

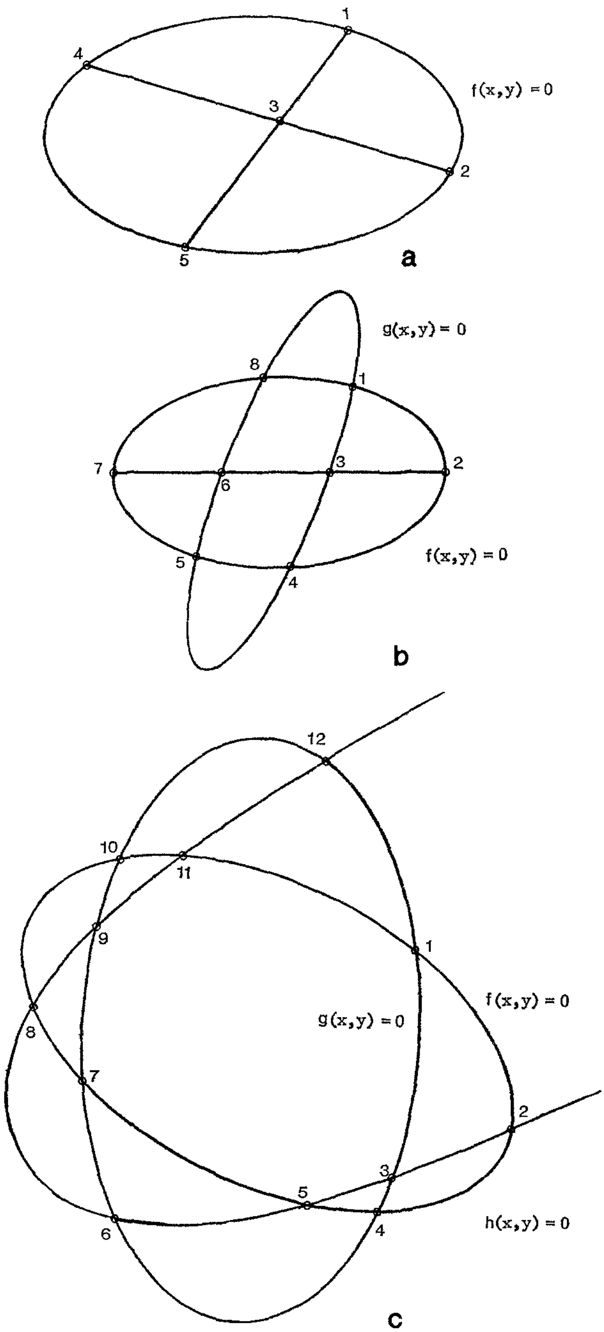

FIG. 2. The completed figures showing the external intersection points for the elements with one conic arc (Fig. 2a), two conic arcs (Fig. 2b), and three conic arcs (Fig. 2c). In each case the element vertices are labeled 1, 2, and 3.

Now, $g(x, y)$, $(8; 1)(4; 5)$, and $f(x, y)$ have a common intersection cycle in pairs and hence,

$$g(x, y) \equiv (8; 1)(4; 5) \bmod f(x, y).$$

Similarly,

$$D(x, y) \equiv (7; 8)(4; 5) \bmod f(x, y);$$

therefore

$$O(x, y)/D(x, y) \equiv (8; 1)(2; 7)/(7; 8) \bmod f(x, y).$$

But

$$(8; 1)(2; 7) \equiv (7; 8)(1; 2) \bmod f(x, y);$$

therefore

$$O(x, y)/D(x, y) \equiv (1; 2) \bmod f(x, y).$$

Similarly, for basis functions associated with nodes on $g(x, y) = 0$, we would have the corresponding

$$O(x, y)/D(x, y) \equiv (1; 3) \bmod g(x, y).$$

3. *Three Conic Sides* (Fig. 2c)

Now $D(x, y)$ is the polynomial associated with the cubic curve through the nine points 4,..., 12.

$$O(x, y) = g(x, y)\, h(x, y).$$

Now

$$\begin{aligned} g(x, y) &\equiv (1; 10)(7; 4) \bmod f(x, y), \\ h(x, y) &\equiv (2; 11)(5; 8) \bmod f(x, y), \\ D(x, y) &\equiv (7; 4)(5; 8)(10; 11) \bmod f(x, y); \end{aligned}$$

therefore

$$O(x, y)/D(x, y) \equiv (1; 10)(2; 11)/(10; 11) \bmod f(x, y).$$

But

$$(1; 10)(2; 11) \equiv (10; 11)(1; 2) \bmod f(x, y);$$

Therefore

$$O(x, y)/D(x, y) \equiv (1; 2) \bmod f(x, y).$$

Similarly, for basis functions WC associated with nodes on $g(x, y) = 0$, we would have

$$O(x, y)/D(x, y) \equiv (1; 3) \bmod g(x, y),$$

and for the ones associated with nodes on $h(x, y) = 0$ we would have

$$O(x, y)/D(x, y) \equiv (2; 3) \bmod h(x, y).$$

Thus in each case

$$O(x, y)/D(x, y) \equiv (\text{linear}) \bmod(\text{conic side}). \tag{13}$$

Now for the basis functions associated with nodes on a conic side we saw that the term $N(x, y)$ was a polynomial of degree $2n$. Hence we arrive at the important result for a node on a conic side $f(x, y) = 0$,

$$WC \equiv P_{2n+1}(x, y) \bmod f(x, y), \tag{14}$$

where $P_{2n+1}(x, y)$ is a polynomial of degree $2n + 1$.

Now returning to Eq. (7), we will use subscripts on the basis functions $\{WC\}, \{WI\}$, and $\{H\}$ to denote the corresponding node, and subscripts on polynomials to denote the degree of the polynomial

$$WI_j + \sum_{i=1}^{m(2n+1)} H_j(x_i, y_i)\, WC_i - H_j(x, y) = P_M(x, y)/D_m(x, y), \tag{15}$$

where $M = 2n + 1 + m$, $j = 1, 2,..., (n/2)(n + 1)$, and $D(x, y)$ has been wirtten $D_m(x, y)$ and is of degree m ($m = 0, 1, 2$ or 3). On each of the element sides $WI_j = 0$ ($j = 1, 2,..., (n/2)(n + 1)$) and we have shown that on element sides

$$WC_i = P_{2n+1}(x, y) \qquad (i = 1, 2,..., m(2n + 1)).$$

Hence

$$P_M(x, y)/D_m(x, y) \equiv Q_{2n+1}(x, y) \bmod(\text{each element side}). \tag{16}$$

On straight sides this is identically zero since the sets $\{WI\}$, $\{WC\}$, and $\{H\}$ have the linear forms of the straight sides as factors. On conic sides this polynomial has simple zeros at $2n + 1$ points of the conic and zeros of multiplicity $n + 1$ at the two vertices concerned. This means that the polynomial is either identically zero or the corresponding algebraic curve of degree $2n + 1$ and the conic have $2n + 1 + 2(n + 1)$, i.e., $4n + 3$, points in common. From Bezout's theorem we see that if they have more than $2(2n + 1)$ then they must have a common component. Since the conic is assumed irreducible, then the polynomial $Q_{2n+1}(x, y)$ must have a factor which is the quadratic form of the conic. This is true for all element sides and hence

$$
\begin{aligned}
P_M(x, y)/D_m(x, y) &\equiv 0,\\
\text{or} \qquad &= S_{m+3}(x, y)\, Q_{2n-2}(x, y)/D_m(x, y),
\end{aligned} \tag{17}
$$

where $S_{m+3}(x, y)$ is the product of the polynomials associated with the element sides.

Now, for each interior node j, the corresponding term $N(x, y)$ was shown to be a polynomial of degree $2n - 2$, which is zero at the remaining $(n/2)(n + 1) - 1$ interior nodes and at the vertices. We now write $N(x, y)$ as $N_{2n-2}(x, y)$ to denote degree. Each of the WC_i is zero at all these points and likewise the H_j functions. Furthermore, all these functions have zeros of multiplicity $n + 1$ at the vertices. Hence $P_M(x, y)/D_m(x, y)$ has simple zeros at $(n/2)(n + 1) - 1$ points on the curve $N_{2n-2}(x, y) = 0$, and zeros of multiplicity $n + 1$ at a further three points of $N_{2n-2}(x, y) = 0$. Hence, if $P_M(x, y)$ is not identically zero, then from Eq. (17), since $S_{m+3}(x, y)$ has no zeros in the interior and exactly double zeros at the vertices, $Q_{2n-2}(x, y)$ must have simple zeros at $(n/2)(n + 1) - 1$ interior points and zeros of multiplicity $n - 1$ at the vertices. But this was exactly the definition of $N_{2n-2}(x, y)$. Therefore

$$P_M(x, y)/D_m(x, y) \equiv 0$$

or

$$= K(S_{m+3}(x, y)\, N_{2n-2}(x, y)/D_m(x, y)), \tag{18}$$

for some constant K.

The proof is completed by noticing that $P_M(x, y)$ is also zero at node j for each $j = 1, 2,..., (n/2)(n + 1)$ and this node, by construction, neither lies on the element sides nor on the corresponding $N_{2n-2}(x, y) = 0$. Therefore

$$P_M(x, y) \equiv 0,$$

i.e.,

$$WI_j + \sum_{i=1}^{m(2n+1)} H_j(x_i, y_i)\, WC_i - H_j(x, y) = 0. \tag{19}$$

Equation (7) is satisfied, the linear system (6) is consistent, and the complete basis is given by Eq. (8).

8. Example

As another example we will show how to construct a basis for up to fifth degree polynomials over an element which has two conic sides and one straight side. For such a degree basis we will be interpolating function value, and first and second partial derivatives at the vertices. We will have three interior nodes and five nodes on each conic side. Such an element is depicted in Fig. 3, where the element vertices are v_1, v_2, and v_3. Let the conic side between v_1 and v_2 be given by $f(x, y) = 0$, that between v_1 and v_3 by $g(x, y) = 0$, and the straight side between v_2 and v_3 by $\ell(x, y) = 0$. We then construct the following polynomials.

Let $PE_i(x, y)$ be the polynomial associated with the algebraic curve defined

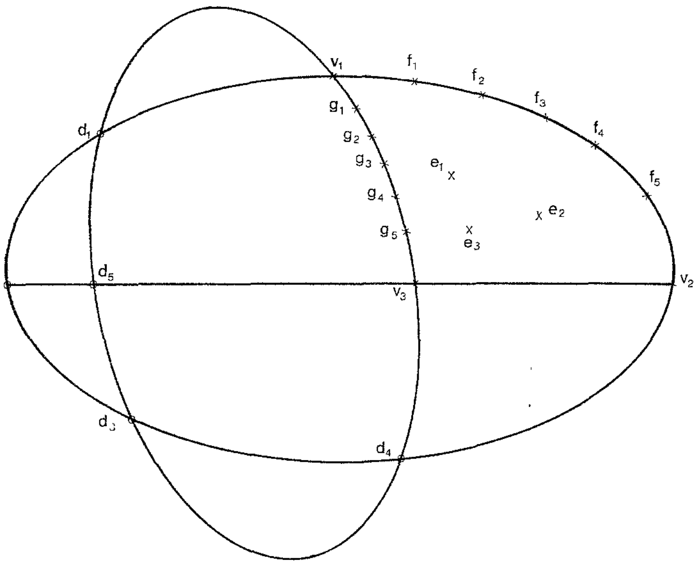

FIG. 3. Node positions for a basis for quintic polynomials on the element with one straight side and two conic sides. This corresponds to the case $n = 2$, $I = 2$ in Eq. (1). The nodes $f_1, ..., f_5$ are on the conic $f(x, y) = 0$, and the nodes $g_1, ..., g_5$ are on $g(x, y) = 0$.

by the points $\{v_j\}$ and $\{e_j\}$, $j \neq i$, which is normalized to have unit value at e_i. Each of these will be quadratic polynomials.

Let $PF_i(x, y)$ be the similarly normalized polynomials associated with the algebraic curve which has simple points at $\{f_j\}$ $(j \neq i)$, $\{e_j\}$, and v_3, and double points at v_1 and v_2. Each of these polynomials will be a quartic.

Similarly, let $PG_i(x, y)$ be the normalized polynomials associated with the algebraic curves, which have simple points $\{g_j\}$ $(j \neq i)$, $\{e_j\}$, and v_2, and double points at v_1 and v_3. These too will be quartics. Let $D(x, y)$ be the quadratic polynomial associated with the curve through the points $\{d_j\}$. There will be ten basis functions associated with nodes on the conic sides, five on each side. We label these WCF_i for the nodes on $f(x, y) = 0$ and WCG_i for the nodes on $g(x, y) = 0$. These will be

$$WCF_i = \alpha_i g(x, y)\, \ell(x, y)\, PF_i(x, y)/D(x, y) \qquad (i = 1, 2, ..., 5), \quad (20)$$

$$WCG_i = \beta_i f(x, y)\, \ell(x, y)\, PG_i(x, y)/D(x, y) \qquad (i = 1, 2, ..., 5). \quad (21)$$

The three basis functions associated with interior nodes will be

$$WI_i = \gamma_i f(x, y)\, g(x, y)\, \ell(x, y)\, PE_i(x, y)/D(x, y) \qquad (i = 1, 2, 3). \quad (22)$$

The α_i, β_i, and γ_i are normalizing constants. The remaining 18 basis functions are then obtained from Eq. (8).

9. Conclusion

Though in many problems it is convenient to consider many small elements and polygonal approximations to curves in the domain, there are problems where it is a distinct advantage to be able to use large elements. This benefit, however, may not be realized if using the large elements incurs a loss of accuracy in approximating the curves a reduction in the order of the bases or errors due to nonconformity. The method of producing bases given here overcomes all these problems and should be suitable for use in problems where large curved elements are desired.

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